4.5 The Dimension of a Vector Space

Theorem 10
If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 11

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Recall the Spanning Set Theorem in $\S\, 4.3$:

Theorem 5. The Spanning Set Theorem
Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in a vector space V, and let $H = \mathrm{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.
a. If one of the vectors in S-say, $\mathbf{v}_k$-is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
b. If $H \neq \{0\}$, some subset of S is a basis for H.

If a nonzero vector space V is spanned by a finite set S, then a subset of S is a basis for V, by the Spanning Set Theorem. In this case, Theorem 11 ensures that the following definition makes sense.

Definition.

If a vector space V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Example 1.

For the given subspace (a) find a basis, and (b) state the dimension.

Rewrite the given vector as:

$$H = \left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\} \qquad a\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} + b\begin{bmatrix}-4\\5\\0\\7\end{bmatrix} + c\begin{bmatrix}-2\\-4\\2\\6\end{bmatrix}$$

Thus $H = \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where $\boxed{\vec{v}_1 = \begin{bmatrix}1\\2\\-1\\-3\end{bmatrix}}$, $\vec{v}_2 = \begin{bmatrix}-4\\5\\0\\7\end{bmatrix}$, $\boxed{\vec{v}_3 = \begin{bmatrix}-2\\-4\\2\\6\end{bmatrix}}$.

Notice that $\vec{v}_3 = -2\vec{v}_1$. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set.

By the Spanning Set Theorem, $\vec{v}_3$ (or $\vec{v}_1$) can be removed.

So $H = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$. Since $\vec{v}_1$ and $\vec{v}_2$ are not multiples of each other. $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

Thus it is a basis for H. So $\dim H = 2$.

Subspaces of a Finite-Dimensional Space

Theorem 12

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Theorem 13 The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

The Dimensions of Nul A, Col A, and Row A

__Definition (Rank, Nullity).__

The **rank** of an $m \times n$ matrix A is the dimension of the column space and the **nullity** of A is the dimension of the null space.

Remark. The rank of an $m \times n$ matrix A is the number of pivot columns and the nullity of A is the number of free variables of $A\vec{x} = \vec{0}$. Since the dimension of the row space is the number of pivot rows, it is also equal to the rank of A.

$\dim \text{Row } A = \dim \text{Col } A = \text{rank } A$

Theorem 14 The Rank Theorem

The dimensions of the column space and the null space of an $m \times n$ matrix A satisfy the equation

$$\text{rank } A + \text{ nullity } A = \text{ number of columns in } A$$

Example 2. Determine the dimensions of $\mathrm{Nul}\,A$, $\mathrm{Col}\,A$, and Row A for the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

Note the matrix A is already in its echelon form.

There are three pivot columns (and rows).

So the dimension of Col A and Row A is 3.

By the rank theorem, dim Nul A = nullity A = 0.

Example 3. The first four Hermite polynomials are $1, 2t, -2+4t^2$, and $-12t+8t^3$. Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

Recall - $\mathbb{P}_3$ is the vector space of polynomials of degree at most 3.

- The standard basis for $\mathbb{P}_3$ is $\{1, t, t^2, t^3\}$ and $\dim \mathbb{P}_3 = 4$.

We are given 4 polynomials, by the Basis Theorem, it's enough to show that they are linearly independent. That is, if

$$x_1 \cdot 1 + x_2 \cdot 2t + x_3 \cdot (-2+4t^2) + x_4 \cdot (-12t+8t^3) = 0 \quad \circledast$$

then the only solution is $x_1 = x_2 = x_3 = x_4 = 0$.

$$\circledast \Rightarrow (x_1 - 2x_3) 1 + (2x_2 - 12x_4) t + (4x_3) t^2 + (8x_4) t^3 = 0$$

This means all the coefficients for $1, t, t^2, t^3$ are zeros.

So we have $\begin{cases} x_1 - 2x_3 = 0 \\ 2x_2 - 12x_4 = 0 \\ 4x_3 = 0 \\ 8x_4 = 0 \end{cases}$ The coefficient matrix is

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

which has 4 pivot position. Thus the only solution for $A\vec{x} = \vec{0}$ is trivial: $x_1 = x_2 = x_3 = x_4 = 0$.

So the given polynomials are linearly independent. and they form a basis for $\mathbb{P}_3$ by the the Basis Theorem.

Example 4. If a 3×8 matrix A has rank 3 , find nullity A, rank A, and $\operatorname{rank} A^T$.

General fact: $\operatorname{rank} A = \operatorname{rank} A^T$

Since: $\operatorname{Col} A^T = \operatorname{Row} A$

$$\begin{array}{ccc} \dim(\operatorname{Col} A^T) & = & \dim(\operatorname{Row} A) \\ \| \text{ by def} & & \| \text{ by } \dim \operatorname{Row} A = \dim \operatorname{col} A = \operatorname{rank} A \\ \operatorname{rank} A^T & = & \operatorname{rank} A \end{array}$$

ANS: nullity A = number of columns of A - $\operatorname{ran} A = 8 - 3 = 5$

$\operatorname{rank} A = 3$.

$\operatorname{rank} A^T = \operatorname{rank} A = 3$.

Rank and the Invertible Matrix Theorem

Theorem. The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\operatorname{rank} A = n$

p. $\operatorname{nullity} A = 0$

q. $\operatorname{Nul} A = \{0\}$

Exercise 5. Find the dimension of the subspace spanned by the given vectors.

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$$

Solution. The matrix A with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are three pivot columns, so the dimension of $\mathrm{Col}\, A$ (which is the dimension of the subspace spanned by the vectors) and Row A is 3 .

Exercise 6. If the nullity of a 7×6 matrix A is 5, what are the dimensions of the column and row spaces of A ?

Solution. $\mathrm{Rank}\, A = 6 - 5 = 1$ so the dimension of the column space and row space is 1 .